

# Octonion X,Y-Product $G_2$ Variants

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## Abstract

The automorphism group  $G_2$  of the octonions changes when octonion X,Y-product variants are used. I present here a general solution for how to go from  $G_2$  to its X,Y-product variant.

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\*\* Happy Birthday to Larry Horwitz. If there's a next time you must come.

## 1. X,Y-Product.

Let  $e_a$ ,  $a = 0, \dots, 7$ , be a basis for the octonion algebra,  $\mathbf{O}$ . There are 7680 distinct ways to define an octonionic multiplication on these 8 symbols such that for all  $a, b \in \{0, \dots, 7\}$  there exists some  $c \in \{0, \dots, 7\}$  satisfying

$$e_a e_b = \pm e_c. \quad (1)$$

My favorite is to let  $e_0 = 1$  be the identity, and let

$$e_a e_{a+1} = e_{a-2}, \quad a \in \{1, \dots, 7\}, \quad (2)$$

and  $a+1, a-2 \in \{1, \dots, 7\}$  are computed modulo 7. See [1 - 4].

Whatever your starting multiplication, it's still true that for all unit elements  $X, Y \in \mathbf{O}$ , and general  $A, B \in \mathbf{O}$ , if we replace the  $\mathbf{O}$  product,  $AB$ , with

$$A \circ_{X,Y} B = (AX)(Y^\dagger B), \quad (3)$$

the resulting algebra is again a copy of the octonions, which I denote  $\mathbf{O}_{X,Y}$ .

In the special case,  $X = Y$ , we denote the resulting copy of the octonions,  $\mathbf{O}_X$  [5]. This product satisfies

$$A \circ_X B = (AX)(X^\dagger B) = X((X^\dagger A)B) = (A(BX))X^\dagger. \quad (4)$$

## 2. Roots.

Any unit octonion can be expressed as

$$X = \exp(\theta x) = \cos \theta + x \sin \theta, \quad (5)$$

where  $x$  is a unit octonion with no real part (so in (5)  $x$  plays a part similar to the complex unit  $i$ ). If  $\theta \neq 0$  or  $\pi$  the  $n^{th}$  roots of  $X$  are easily defined:

$$X^{\frac{1}{n}} = \exp\left[\left(\frac{\theta + 2\pi k}{n}\right)x\right], \quad k = 0, \dots, n-1. \quad (6)$$

Although the  $n^{th}$  roots of  $X$  are not uniquely defined, it is clear from (6) that as long as  $X \neq \pm 1$ , then  $X$  and all its  $n^{th}$  roots commute. Consequently any product of three octonions, two of which are  $n^{th}$  roots of  $X$ , must associate. (If  $X = -1$ , for example, then the square roots of  $X = -1$  are all the elements of the 6-sphere of unit octonions with no real part.)

### 3. $G_{2X}$ .

Therefore, given a unit octonion  $X \neq \pm 1$ , a given cube root, arbitrary  $A, B \in \mathbf{O}$ , and using (4),

$$\begin{aligned}
A \circ_X B &= (AX)(X^\dagger B) \\
&= (AX^{2/3}X^{1/3})(X^{-1/3}X^{-2/3}B) \\
&= X^{1/3}[(X^{-1/3}AX^{2/3})(X^{-2/3}B)] \\
&= X^{1/3}[(X^{-1/3}AX^{1/3}X^{1/3})(X^{-1/3}X^{-1/3}B)] \\
&= X^{1/3}\{[(X^{-1/3}AX^{1/3})(X^{-1/3}BX^{1/3})]X^{-1/3}\} \\
&= X^{1/3}[(X^{-1/3}AX^{1/3})(X^{-1/3}BX^{1/3})]X^{-1/3}.
\end{aligned} \tag{7}$$

Note that if  $X = \pm 1$ , then  $A \circ_X B = AB$ , in which case (7) implies

$$X^{-1/3}(AB)X^{1/3} = (X^{-1/3}AX^{1/3})(X^{-1/3}BX^{1/3}). \tag{8}$$

That is, if  $U$  is a  $6^{th}$  root of unity, then the action  $A \rightarrow UAU^\dagger$  is an element of  $G_2$ , the automorphism group of  $\mathbf{O}$ .

From now on let  $G_2$  denote the automorphism group of our original copy of  $\mathbf{O}$ , and  $G_{2X,Y}$  and  $G_{2X}$  the automorphism groups of  $\mathbf{O}_{X,Y}$  and  $\mathbf{O}_X$  (all copies of  $\mathbf{O}$  are isomorphic, of course). Let  $LG_2$ ,  $LG_{2X,Y}$  and  $LG_{2X}$  be their respective Lie algebras.

For any  $A \in \mathbf{O}$ , define  $A_L$  and  $A_R$  mapping  $\mathbf{O} \rightarrow \mathbf{O}$  by

$$A_L[B] = AB, \quad A_R[B] = BA. \tag{9}$$

Let  $g \in LG_2$ , which satisfies

$$g[AB] = (g[A])B + A(g[B]).$$

Therefore, if  $X \neq \pm 1$  is a unit octonion, and  $X^{1/3}$  a chosen cube root, then

$$\begin{aligned}
X_R^{-1/3} X_L^{1/3} g X_L^{-1/3} X_R^{1/3} [A \circ_X B] &= X_R^{-1/3} X_L^{1/3} g [X^{-1/3} (A \circ_X B) X^{1/3}] \\
&= X_R^{-1/3} X_L^{1/3} g [(X^{-1/3} A X^{1/3}) (X^{-1/3} B X^{1/3})] \\
&= X_R^{-1/3} X_L^{1/3} [g [X^{-1/3} A X^{1/3}] (X^{-1/3} B X^{1/3}) + (X^{-1/3} A X^{1/3}) g [X^{-1/3} B X^{1/3}]] \\
&= X_R^{-1/3} X_L^{1/3} [g X_L^{-1/3} X_R^{1/3} [A] (X^{-1/3} B X^{1/3}) + (X^{-1/3} A X^{1/3}) g X_L^{-1/3} X_R^{1/3} [B]] \\
&= X_R^{-1/3} X_L^{1/3} [(X^{-1/3} (X_R^{-1/3} X_L^{1/3} g X_L^{-1/3} X_R^{1/3} [A]) X^{1/3}) (X^{-1/3} B X^{1/3}) \\
&\quad + (X^{-1/3} A X^{1/3}) (X^{-1/3} (X_R^{-1/3} X_L^{1/3} g X_L^{-1/3} X_R^{1/3} [B]) X^{1/3})] \\
&= (X_R^{-1/3} X_L^{1/3} g X_L^{-1/3} X_R^{1/3} [A]) \circ_X (B) + A \circ_X (X_R^{-1/3} X_L^{1/3} g X_L^{-1/3} X_R^{1/3} [B]).
\end{aligned} \tag{10}$$

Therefore,

$$\begin{aligned}
L G_{2X} &= X_R^{-1/3} X_L^{1/3} L G_2 X_L^{-1/3} X_R^{1/3} \\
G_{2X} &= X_R^{-1/3} X_L^{1/3} G_2 X_L^{-1/3} X_R^{1/3}.
\end{aligned} \tag{11}$$

#### 4. $G_{2(1,Z)}$ .

The next step to a completely general solution to  $G_{2(X,Y)}$  is the case  $G_{2(1,Z)}$ ,  $Z \neq \pm 1$ , with the altered product

$$\begin{aligned}
A \circ_{1,Z} B &= A(Z^\dagger B) \\
&= (A Z^{-2/3} Z^{2/3}) (Z^{-2/3} Z^{-1/3} B) \\
&= Z^{2/3} ((Z^{-2/3} A Z^{-2/3}) (Z^{-1/3} B)) \\
&= Z^{2/3} ((Z^{-2/3} A Z^{-1/3} Z^{-1/3}) (Z^{1/3} Z^{-2/3} B)) \\
&= Z^{2/3} ((Z^{-2/3} A Z^{-1/3}) (Z^{-2/3} B Z^{-1/3})) Z^{1/3}
\end{aligned} \tag{12}$$

Therefore, in much the same way as was done above we prove

$$\begin{aligned}
L G_{2(1,Z)} &= Z_R^{1/3} Z_L^{2/3} L G_2 Z_L^{-2/3} Z_R^{-1/3}, \\
G_{2(1,Z)} &= Z_R^{1/3} Z_L^{2/3} G_2 Z_L^{-2/3} Z_R^{-1/3}.
\end{aligned} \tag{13}$$

## 5. $G_{2(X,Y)}$ .

Finally, as I have noted elsewhere [4],

$$A \circ_{X,Y} B = (AX)(Y^\dagger B) = A \circ_X (Z^\dagger \circ_X B), \quad (14)$$

where  $Z = YX^\dagger$  is the identity of the X,Y-product. Anyway, using the results above,

$$\begin{aligned} (AX)(Y^\dagger B) &= A \circ_X (Z^\dagger \circ_X B) \\ &= Z^{2/3} \circ_X ((Z^{-2/3} \circ_X A \circ_X Z^{-1/3}) \circ_X (Z^{-2/3} \circ_X B \circ_X Z^{-1/3})) \circ_X Z^{1/3} \\ &= X^{1/3} \{ (X^{-1/3} Z^{2/3} X^{1/3}) [ < (X^{-1/3} Z^{-2/3} X^{1/3}) (X^{-1/3} A X^{1/3}) (X^{-1/3} Z^{-1/3} X^{1/3}) > \bullet \\ &< (X^{-1/3} Z^{-2/3} X^{1/3}) (X^{-1/3} B X^{1/3}) (X^{-1/3} Z^{-1/3} X^{1/3}) > ] (X^{-1/3} Z^{1/3} X^{1/3}) \} X^{-1/3}, \end{aligned} \quad (15)$$

implying in the way that (11) and (13) were derived that

$$\begin{aligned} LG_{2X,Y} &= \\ &X_R^{-1/3} X_L^{1/3} (X^{-1/3} Z^{1/3} X^{1/3})_R (X^{-1/3} Z^{2/3} X^{1/3})_L \bullet \\ &LG_{2\bullet} \\ &(X^{-1/3} Z^{-2/3} X^{1/3})_L (X^{-1/3} Z^{-1/3} X^{1/3})_R X_L^{-1/3} X_R^{1/3}, \end{aligned} \quad (16)$$

$$\begin{aligned} G_{2X,Y} &= \\ &X_R^{-1/3} X_L^{1/3} (X^{-1/3} Z^{1/3} X^{1/3})_R (X^{-1/3} Z^{2/3} X^{1/3})_L \bullet \\ &G_{2\bullet} \\ &(X^{-1/3} Z^{-2/3} X^{1/3})_L (X^{-1/3} Z^{-1/3} X^{1/3})_R X_L^{-1/3} X_R^{1/3}. \end{aligned}$$

Note that if  $X = Y$ , so  $Z = 1$ , then (16) reduces to (11) if we choose  $Z^{2/3} = 1$ , and if  $X = 1$ , and we choose  $X^{1/3} = 1$ , then (16) reduces to (13). However, different roots of unity could be chosen leading to variations of (11) and (13).

The result (16) is in no way unique. I have found at least one very different looking solution, but none with the same symmetric appearance as (16).

## 6. Conclusion.

As usual, I would emphasize that my motivation for pursuing this arcane line of research is to gain an insight into the workings of the mathematics, and ultimately to use that insight to light a path to its further relationship to physics [1].

## References

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